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DYNAMICS OF VARIABLE SYSTEMS AND LIE GROUPS†

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Mechanical systems whose configuration space is a Lie group and whose Lagrangian is invariant to left translations on that group are considered. It is assumed that the mass geometry of the system may change under the action of only internal forces. The equations of motion admit of a complete set of Noether integrals which are linear in the velocities. For fixed values of these integrals, the equations of motion reduce to a non-autonomous system of first-order differential equations on the Lie group. Conditions under which the system can be brought from any initial position to another preassigned position by changing its mass geometry are discussed. The "falling cat" problem and the problem of the motion of a body of variable shape in an unlimited volume of ideal fluid are considered as examples.

1. VARIABLE SYSTEMS ON LIE GROUPS

Let G be a connected n-dimensional Lie group which is the configuration manifold of a mechanical system with $n > 1$ degrees of freedom. Let g be its Lie algebra – an n-dimensional vector space with the natural commutation operation [,]. For example, we may assume that g is the linear space of left-invariant vector fields on the group G, the operation [,] corresponding to the usual commutator (Jacobi bracket) of tangent vector fields.

Let $x = (x_1, \dots, x_n)$ be local coordinates on G, and let $\dot{x} = (\dot{x}_1, \dots, \dot{x}_n)$ be the velocity of the system – the tangent vector to G at the point x. We introduce a basis $\upsilon_1, \dots, \upsilon_n$ of independent left-invariant vector fields (they are linearly independent at all points of G). The velocity $\dot{x}$ may be expressed as a linear combination of basis elements

$$\dot{x} = \sum \omega_k \upsilon_k \tag{1.1}$$

The coefficients $\omega = (\omega_1, \dots, \omega_n)$ depend on x and $\dot{x}$, the dependence being linear with respect to the velocities. They are known as quasi-velocities and serve as coordinates on the algebra g.

We will consider inertial motion, so that the Lagrangian reduces to the kinetic energy $T = T_2 + T_1 + T_0$. We shall assume that the function T is left-invariant; in other words, it depends only on the quasi-velocities ω and the time t. The time dependence is due to possible changes in the mass geometry of the system solely owing to internal forces.

Thus,

$$T_2 = (I\omega, \omega)/2, \quad T_1 = (\lambda, \omega) \tag{1.2}$$

where the inertia matrix $I = \|I_{ij}\|$ and the gyroscopic vector $\lambda = (\lambda_1, \dots, \lambda_n)$ are previously known functions of time. Since the free term T_0 depends only on time, it is not essential (since it does not enter into the equations of motion). The Poincaré equations will be equations only on the Lie algebra g

$$\left(\frac{\partial T}{\partial \omega_k}\right)^{\cdot} = \sum c_{ki}^j \omega_i \frac{\partial T}{\partial \omega_j}, \quad k = 1, \dots, n. \tag{1.3}$$

where c_{ki}^j are the structure constants of the algebra g. For details see, e.g. [1, Chap. III].

We present two examples.

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1. *Liouville's problem of the rotation of a variable body* [2]. We associate with the body fixed axes which, at each instant of time, will be the principal axes of inertia of the body; the origin of the moving system coincides with the fixed centre of mass. We shall assume that the mass geometry of the system of points may vary under the action of internal forces according to a prescribed law. Under these assumptions, Eqs (1.3) become Liouville's equations

$$(I\omega+\lambda)^{\cdot}+\omega\times(I\omega+\lambda) = 0 \tag{1.4}$$

where $I = \operatorname{diag}(I_1, I_2, I_3)$ and ω is the angular velocity of rotation of the moving trihedron. The vector $I\omega + \lambda$ is the angular momentum of the variable body about its centre of mass. Equations (1.4) have been extended [3] to the case of the three-dimensional motion of a variable body (when $G = E(3)$).

2. *The problem of the motion of a variable body in an infinite volume of ideal fluid which is in irrotational motion and at rest at infinity* [3]. In this case G is $E(3)$ – the group of motions of Euclidean three-space, and Eqs (1.3) become Kirchhoff's system of equations

$$\left(\frac{\partial T}{\partial\omega}\right)^{\cdot}+\upsilon\times\frac{\partial T}{\partial\upsilon} = 0,\quad \left(\frac{\partial T}{\partial\upsilon}\right)^{\cdot}+\omega\times\frac{\partial T}{\partial\omega}+\upsilon\times\frac{\partial T}{\partial\upsilon} = 0 \tag{1.5}$$

where υ is the velocity of the origin of the moving frame of reference, ω is the angular velocity of the moving trihedron, and T is the kinetic energy of the system body + fluid, which depends on υ, ω and the time t.

2. THE DYNAMICS OF VARIABLE SYSTEMS WITH ZERO ANGULAR MOMENTUM

Variable systems on Lie groups with left-invariant kinetic energy may be investigated by a method developed in [1, Chap. III]. To that end, let us consider n independent *right-invariant* vector fields $w_1, \dots, w_n$. The phase flows that they generate are families of *left* translations on G. Since the Lagrangian (which is identical with the kinetic energy) is by assumption left-invariant, the complete system of differential equations (1.1), (1.3) admits of n independent Noether integrals

$$\frac{\partial T}{\partial\omega}\cdot w_1 = c_1, \dots, \frac{\partial T}{\partial\omega}\cdot w_n = c_n \tag{2.1}$$

where $c_1, \dots, c_n$ is a sequence of arbitrary constants. The left-hand sides of these equations are linear in the velocities $\dot{x}_1, \dots, \dot{x}_n$. Since the quadratic form T_2 is non-degenerate, system (2.1) is solvable for the velocities

$$\dot{x} = \upsilon(x, t, c),\quad x\in G \tag{2.2}$$

This is a system of equations on the group depending on the parameters c. It has been shown [1] that the phase flow (2.2) has properties analogous to those of flows in a multidimensional ideal fluid.

We will consider the case in which $c = 0$; we may assume that the system is initially at rest, and then, under the action of internal forces, its mass geometry begins to change. Equations (2.2) will then be considerably simplified. Indeed, since $c = 0$ and the vector fields $w_1, \dots, w_n$ are linearly independent at all points of G, it follows that $\partial T/\partial\omega = 0$. By formulae (1.2)

$$\frac{\partial T}{\partial\omega} = I\omega+\lambda = 0$$

Consequently, the velocity $\omega = -I^{-1}\lambda$ is a known function of time.

We may assume without loss of generality that the inertia tensor has been reduced to diagonal form: $I = \operatorname{diag}(I_1, \dots, I_n)$. If $\lambda_1, \dots, \lambda_n$ are the components of the covector of gyroscopic forces λ, then

$$\omega_k = -\frac{\lambda_k}{I_k},\quad k = 1, \dots, n$$

and consequently Eqs (2.2) for $c = 0$ follow at once from system (1.1)

$$\dot{x} = -\sum_{k=1}^{n} \frac{\lambda_k}{I_k} v_k(x) \tag{2.3}$$

We recall that $v_1, \ldots, v_n$ is a fixed sequence of independent vector fields on the group G.

The non-autonomous system (2.3) has important properties. For example it admits of an integral invariant (invariant measure) whose density is identical with the density of a right-invariant measure on G. This result is derived from the results of considering the stationary case [4]. In particular, for a unimodular group G (this includes, for example, all compact groups), the flow of system (2.3) preserves a two-sided Haar measure.

On the other hand, the flow of system (2.3) transfers integral curves of every right-invariant vector field w to integral curves of the same field. In other words, these curves are frozen into the flow of system (2.3). Indeed, the condition for integral curves of a stationary field $w(x)$ to be frozen has the form $[w, v] = \mu w$, where μ is some function of x and t (a simple proof of this condition may be found in [5]). But $[w, v_k] = 0$ for all k, since the phase flows generated by left-invariant (right-invariant) vector fields on a Lie group are families of right (left) translations.

Example 1. In Liouville's problem of the rotation of a variable body, Eqs (2.3) have the form

$$\begin{aligned} &\dot{\vartheta} = -\frac{\lambda_1 \cos\varphi}{I_1} + \frac{\lambda_2 \sin\varphi}{I_2}, \quad \dot{\varphi} = \frac{\lambda_1 \sin\varphi\cos\theta}{I_1 \sin\theta} + \frac{\lambda_2 \cos\varphi\cos\theta}{I_2 \sin\theta} - \frac{\lambda_3}{I_3} \\ &\dot{\psi} = -\frac{\lambda_1 \sin\varphi}{I_1 \sin\theta} - \frac{\lambda_2 \cos\varphi}{I_2 \cos\theta} \end{aligned} \tag{2.4}$$

where ϑ, φ and ψ are the Euler angles – coordinates on the group $SO(3)$. According to system (2.3), and the coefficients of $-\lambda_k/I_k$ in these formulae define the components of independent left-invariant vector fields on the group $SO(3)$. Indeed, for example

$$\cos\varphi, \quad -\sin\varphi\cos\theta/\sin\theta, \quad \sin\varphi/\sin\theta \tag{2.5}$$

are the components of unit angular velocity directed along the first moving axis (this follows from the kinematic Euler equations). But this immediately implies that (2.5) are the components of a left-invariant vector field.

Equations (2.4) admit of an integral invariant

$$\iiint \sin\theta d\theta d\varphi d\psi \tag{2.6}$$

which is identical with a two-sided invariant Haar measure on the group $SO(3)$ (see, e.g. [6]). In Euler angles ϑ, φ, ψ, the right-invariant field corresponding to rotation of a trihedron at unit angular velocity about the third fixed axis has components 0, 0, 1. Consequently, the integral curves of this field are given by the equations ϑ, φ = const. Since the equations of system (2.4) do not explicitly contain in the angle ψ, this implies that the aforementioned curves are frozen into the flow of system (2.4).

Note that Eqs (2.4) also hold for a gyrostat – a rigid body with symmetrical flywheels. The mass distribution of such a body obviously does not change (I_k = const) and the gyroscopic moment $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is also constant. In that case Eqs (2.4) can be integrated explicitly. Since system (2.4) has the integral invariant (2.6), it is sufficient for its integrability to know a first integral which is not a constant. The existence of such an integral is an exceptional phenomenon.

Equations (2.4) may be expressed as a linear system of differential equations with redundant variables. Indeed, it follows from Liouville's equations (1.4) that the angular momentum vector $K = I\omega + \lambda$ maintains a fixed value and direction in the fixed space. By assumption, $K = 0$. Consequently, $\omega = -I^{-1}\lambda$. Let α, β, γ be a fixed orthonormal frame of reference. Then Poisson's equations

$$\dot{\alpha} + \omega \times \alpha = 0, \quad \dot{\beta} + \omega \times \beta = 0, \quad \dot{\gamma} + \omega \times \gamma = 0 \tag{2.7}$$

will be a closed system of linear equations with variable coefficients. The nine components of the vectors α, β and γ obey six orthogonality relations. It is interesting to note that, for system (2.7) to be integrable, it is sufficient to know just one non-trivial solution $\xi(t)$ of Poisson's equations. Indeed, the function (ξ, α) will be a first integral of the first of equations (2.7)

$$(\xi, \alpha)^{\cdot} = (\xi \times \omega, \alpha) + (\xi, \alpha \times \omega) = 0$$

These remarks may be generalized. By Ado's theorem, every finite-dimensional Lie algebra admits of a representation in a finite-dimensional vector space. Consequently, Eqs (2.3) may also be represented as a linear system of differential equations (with a redundant sequence of variables).

Example 2. Let us consider the special case of the plane-parallel motion of a variable body in a fluid without the action of external forces: the body moves in such a way that at every instant of time its shape and mass distribution are symmetrical about a certain fixed plane Π. Let x and y be the Cartesian coordinates of the origin of the moving frame of reference in the plane Π, and let α be the angle of rotation of the moving axes. It is well known (see [3]) that, at all times, the moving frame of reference may be chosen in such a way that the kinetic energy of the body + fluid system has the form

$$T = (a_1 v_1^2 + a_2 v_2^2 + b\omega^2)/2 + \lambda_1 v_1 + \lambda_2 v_2 + \mu\omega + \chi$$

where $\omega = \dot{\alpha}$ is the angular velocity of the moving frame, and v_1 and v_2 are the projections of the velocity of the origin of that frame on the moving axes. The coefficients in this formula are assumed to be known functions of time. In the case of plane-parallel motion of the body, G is the group of motions of the plane $E(2)$.

Equations (2.3) become

$$\dot{x} = -\frac{\lambda_1 \cos\alpha}{a_1} + \frac{\lambda_2 \sin\alpha}{a_2}, \quad \dot{y} = -\frac{\lambda_1 \sin\alpha}{a_1} - \frac{\lambda_2 \cos\alpha}{a_2}, \quad \dot{\alpha} = -\frac{\mu}{b} \tag{2.8}$$

where it is assumed, of course, that the total momentum and angular momentum of the body + fluid system are zero.

We will indicate left-invariant vector fields on $E(2)$

$$X = (\cos\alpha, -\sin\alpha, 0), \quad Y = (\sin\alpha, \cos\alpha, 0), \quad Z = (0, 0, 1)$$

Their commutators are

$$[X, Y] = 0, \quad [X, Z] = Y, \quad [Y, Z] = -X \tag{2.9}$$

Unlike system (2.4), Eqs (2.8) are easily integrated in the most general case. The flow of system (2.8) conserves the standard measure on the group $E(2) = \{x, y, \alpha, \mathrm{mod} 2\pi\}$.

3. THE CONDITIONS FOR COMPLETE CONTROLLABILITY

Putting $u_k = -\lambda_k/I_k$ in system (2.3), let us treat these functions as controls. More precisely, let us assume that $u_k \equiv 0$ for $k > m$, and $u_1, \dots, u_m$ are arbitrary piecewise-smooth functions of time satisfying the inequalities $|u_k(t)| \le \varepsilon$. The question is whether the m controls can be chosen in such a way that the system will go from any initial position $x^1 \in G$ to any prescribed position $x^2 \in G$. This property is known as *complete controllability*.

Obviously, if $m = 1$, the system cannot be completely controllable: it cannot coincide with an integral curve of the left-invariant field v_1, which in turn can never fill all of G (of course, if $n > 1$). The question of complete controllability becomes more interesting for $m = 2$.

Theorem. The system

$$\dot{x} = \sum_{k=1}^{m} u_k(t) v_k(x), \quad x \in G \tag{3.1}$$

is completely controllable if and only if the fields $v_1, \dots, v_m$ are not contained in any subalgebra of g other than g itself.

This condition may be reformulated as follows. Let us assume that, among the vector fields $v_1, \dots, v_m$ and the vector fields obtained from them by successive applications of the commutation operation, one can find n vector fields $V_1, \dots, V_n$ which are linearly independent at least at one point of G. Then from any point of the connected group G one can move to any other, moving a finite number of times over trajectories of the fields $\upsilon_1, \dots, \upsilon_m$. This is the content of a well-known theorem of Rashevskii–Chow [7, 8]. Since the fields $V_1, \dots, V_n$ are left-invariant (as commutators of left-invariant fields on the group G), the fact that they are linearly independent at least at one point of G implies that they are linearly independent at all points of G. The required control is constructed in such a way that the time interval is divided into intervals Δ_k in which all controls except one, say $u_k(t)$, vanish, while $u_k(t) = \varepsilon$ or $u_k(t) = -\varepsilon$ for $t \in \Delta_k$.

The sufficient condition just formulated for complete controllability is also necessary. Indeed, let the fields $\upsilon_1, \dots, \upsilon_m$ generate a proper subalgebra $g' \subset g$, $\dim g' = m' < n$. The algebra g' contains all left-invariant vector fields $V_1, \dots, V_{m'}$ obtained from the fields $\upsilon_1, \dots, \upsilon_m$ by successive application of the commutation operation. Let us define on G an m'-dimensional distribution of tangent vectors, generated at each point $x \in G$ by linear combinations of vectors $V_1(x), \dots, V_{m'}(x)$. Since the subalgebra g' is closed under commutation, this distribution is integrable. Consequently, the whole group is stratified into a family of integral manifolds Σ_c (where c is a sequence of $n - m'$ independent parameters) such that the tangent space to Σ_c at a point x is a linear combination of vectors $V_1(x), \dots, V_{m'}(x)$. Consequently, if x_0 is a point in some space Σ_c, then for all $u(t), \dots, u_m(t)$ the solution of system (3.1) with initial condition x_0 is also in Σ_c. It remains to note that, since $\dim\Sigma_c = m' < n$, Σ_c cannot be the whole group G. This completes the proof.

Since a commutator of left-invariant vector fields is a linear combination of $\upsilon_1, \dots, \upsilon_n$ with constant coefficients, the above problem of complete controllability is purely algebraic. Moreover, the complete controllability conditions depend exclusively on the group structure of G (of course, after the vector fields $\upsilon_1, \dots, \upsilon_n$ have been chosen). In particular, if the group G is commutative, complete controllability is possible only if $m = n$.

As already remarked (Section 2), using redundant coordinates, one can represent system (3.1) as a linear system of differential equations. This enables the well-developed theory of optimal control to be applied in the linear case (see [9]).

Example 1. Equations (2.4) describe the rotation of a rigid body with symmetric flywheels, where the rotation of the flywheels (gyrodynes) may be controlled by internal forces. Then $I_k = \text{const}$, and it is natural to choose as controls the relative angular momenta λ_k of the flywheels. This problem has been extensively studied (see, e.g. [10]). In particular, controlling only two flywheels (not on the same axis), one can rotate the body from any position into another. This is a simple corollary of the theorem (making use of the properties of the group $SO(3)$).

If one puts $u_k = -\lambda_k/I_k$ and now takes the functions u_k as controls, this case reduces to the aforementioned problem of a body with flywheels. The method developed here may also be applied to the well-known *falling cat problem* (the first publication of the theorem [11] was accompanied by photographs showing how a falling cat reverses its orientation in space). To explain the effect of the change of orientation, the cat is often simulated by Lagrange gyroscopes linked together by hinges [12, 13]. This approach makes it necessary to deal with a fairly complex dynamical system, whose configuration space is the direct product $SO(3) \times SO(3)$. The approach proposed here reduces the problem essentially to the problem of a body with flywheels. Following [12, 13], one can consider the *time-optimal problem*, which has been studied in detail for a body with flywheels [10].

Note that, in the falling cat problem, attention may be confined to the system consisting of the first two equations (2.4) (since the cat's angle of rotation when it lands is of no interest). Here is a simple example of guaranteed feedback control

$$u_1 = \xi\cos\varphi, \quad u_2 = -\xi\sin\varphi, \quad u_3 = \eta,$$

where ξ and η are arbitrary continuous functions of time with $\xi(t) > 0$. The first equation of system (2.4) takes the form $\dot{\vartheta} = \xi$, and therefore the angle ϑ changes in a finite time from 0 to π, which solves the control problem.

Example 2. If one puts $u_k = -\lambda_k/a_k$ $(k = 1, 2)$ and $u_3 = -\mu/b$ in system (2.7) and puts $u_3 = 0$, the variable "plane" body + fluid system will not be completely controllable (according to Eqs (2.9)). But if one puts $u_1 = 0$ $(u_2 = 0)$, then a suitable choice of controls u_2, u_3 (or u_1, u_3) makes it possible to steer the moving frame from any position to any prescribed position.

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